Monotonicity and robustness of majority rule

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ARTICLE INFO

Article history:
Received 3 October 2009
Received in revised form 20 January 2010
Accepted 25 February 2010
Available online 2 March 2010

JEL classification:
D71

Keywords:
Majority rule
Monotonicity
Well-working
Continuum number of voters
Diverse domain

ABSTRACT

I show that the majority rule is not “superior” to other rules if independence of irrelevant alternatives is replaced with monotonicity in the Dasgupta and Maskin (2008a) framework. In addition, I introduce a diversity requirement for preferences that restores the superiority of the majority rule in case of monotonicity.

1. Introduction

Research on Nash implementability of social choice rules on restricted domains is rare in the literature. The appropriate domains, where they are Nash implementable, for the plurality and Borda rule were carefully analyzed (Sanver, 2009; Puppe and Tasnádi, 2008). Both papers search for domains on which the specific rule is monotonic, thus Nash-implementable by Maskin (1999).

Dasgupta and Maskin (2008a) introduce a framework in which different rules can be compared in terms of satisfying simple conditions (anonymity, neutrality, Pareto property, independence of irrelevant alternatives and generic decisiveness) defining whether a rule is well-working on a restricted domain. They found that the majority rule is the best possible rule as it works well on each domain where any other rule works well. They also examine strategic voting in a further work (Dasgupta and Maskin, 2008b).

In this paper I examine social choice rules in terms of monotonically well-working. I follow the framework of Dasgupta and Maskin (2008a) (continuum number of voters and Borel-profiles), but define monotonically well-working by picking monotonicity instead of independence of irrelevant alternatives. I show that major rule is not superior (there are specific domains where it is outperformed by the Borda rule). Furthermore, I give a requirement for the set of possible preferences such that it works monotonically well if any other rule does so. This requirement is characterized by containing orders of each permutation cycle for any triplet of alternatives.

Section 2 describes the framework, the definitions for social choice rules and properties, the characteristics of the majority rule and the Borda rule on the full domain. Section 3 contains two propositions (the first states that majority rule is not monotonically superior, the second that in case of diverse domains, however, it is) and their proofs.

2. Notations

A continuum of voters indexed by [0, 1] decide over the finite set of alternatives A, |A| ≥ 3. Each voter has a strict preference on the elements of A. Let D_A stand for the set of all strict orderings on A and D for the restricted domain, where D ⊂ D_A.

Let us call the function P: [0, 1] → D a profile, where P(i) is the preference of voter i. Let us write xP(i)-overlay if x is preferred to y. Let Q be the set of all possible profiles with respect to D. Let P_1 be the restriction of profile P to the subset Y ⊆ A. We consider only those profiles for which the sets {i: xP(i)y} are measurable with respect to the Lebesgue-measure for all x, y ∈ A. Let us refer to them as Borel-profiles. Let μ denote the Lebesgue-measure and quantile P(Y) the measure of set {i: xP(i)y}.

Let us call the set-valued function F: Q × P(Y) → P(Y) a social choice rule, which is endowed by the following properties: F(P, Y) ⊆ Y for all pair (P, Y), and P_{Y} = P_1[Y], yields F(P, Y) = F(P, Y). Let F be the majority

☆☆ This research is part of my Master's Thesis defended at Corvinus University of Budapest, Hungary.
☆☆☆ I am very grateful to Attila Tasnádi for his guidance and suggestions as my advisor. E-mail address: omariann@gmail.com.
rule, where \( F^M(P, Y) = \{ x \in Y : q_\alpha(y, x) \geq 1, \forall z \in Y \} \). Let \( F^B \) denote the Borda rule, where \( F^B(P, Y) = \{ x \in Y : \frac{1}{3} \sum_{i=1}^{n} x_i \geq \frac{1}{3} \sum_{i=1}^{n} Y_i \} \). We call a social choice rule \( F \) anonymous (A) if a measure preserving permutation on the set of voters \( \sigma \in [0, 1] \) does not alter the outcome, i.e. \( F(P^\sigma, Y) = F(P, Y) \). We call \( F \) neutral (N) if the permutation on the alternatives, \( \sigma: Y \rightarrow Y \) yields \( F(P^\sigma, Y) = \sigma F(P, Y) \). \( F \) satisfies the Pareto property (P) if \( x, y \in A, x \neq y, x \in Y, xP(y) \) for all \( i \), i.e. \( y \in F(P, Y) \). \( F \) satisfies the independence of irrelevant alternatives (IIA) if \( x \in F(P, Y), x \in Y \rightarrow \sigma x \in F(\sigma P, Y) \).

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In Dasgupta and Maskin (2008a), the social choice rules, which satisfy A, N, P, IIA and GD on a domain, are said to work well on that domain. In any case when \( F \) is also GD on \( A, N, P \), it can be easily seen, that the majority rule is A, N, M, P, IIA on any domain. Furthermore, it does not satisfy IIA, M, GD.\(^7\)

3. The results

**Proposition 3.1.** The majority rule is not superior among social choice rules in terms of working monotonically well.

**Proof.** First, I show that the Borda rule outperforms the majority rule on the domain \( D \), in which all the preferences start with one of the three lists, \( (x, y, z), (y, z, x) \) or \( (z, x, y) \).

As \( D \) contains a cycle, the majority rule works not monotonically well on this domain. Contrariwise, the Borda rule does so. As we discussed above, the Borda rule is A, N, P on any domain. Furthermore, it can be easily seen, that it is M on \( D \).

Second, I show that the Borda rule is also GD on \( D \). Let us pick a general profile \( y \leq x \leq z \).

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
 x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

where \( \alpha, \beta, \gamma \) denote the measure of the set of voters who are endowed by the respective preferences. In any case when \( \alpha, \beta, \gamma \) satisfy the triangle inequality, then none of the three alternatives beat the others, thus \( F^B(P, (x, y, z)) = \emptyset \) for too many possible ratios of sets of voters.

A simple reasoning, why the majority rule cannot be GD on \( D \) with a cycle, is that if we have a profile \( P \) sketched below

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
 x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

In any case of \( 3\alpha = \beta \), \( F^B(P, (x, y, z)) \) is not single.

The Borda rule chooses either a single alternative or, in case of equal scores, more from these three alternatives. Let us pick the pair \( (x, y) \) first. Both of them would be chosen if \( \alpha + \gamma = 2\beta \), i.e. \( \beta = \frac{1}{2} \) because of \( \alpha + \beta + \gamma = 1 \). Which means \( \frac{3\alpha(x, y)}{D(x, y)} = 2 \) must be ruled out with help of \( S \). In other cases, we cannot get equal scores for \( x \) and \( y \). By examining any other pairs, we have to rule out the same ratio. Thus the Borda rule is GD with \( S = \{x, y \} \).

**Definition 3.2.** Let us call a domain diverse if it contains preferences from both of the permutation cycles\(^8\) of any three alternatives.

**Lemma 3.3.** Let us consider a social choice rule \( F \), which works monotonically well on a domain \( D \) containing a permutation cycle of \( x, y, z \). If \( F(P, (x, y, z)) \) is a singleton for the following \( P = \)

\[
\begin{array}{ccc}
\alpha & \beta & \gamma - \alpha \\
 x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

then \( F(P, (x, y, z)) = z \).

**Proof.** Let us suppose indirectly, that \( F(P, (x, y, z)) = x \). The following profile \( P \) is also possible on \( D \):

\[
\begin{array}{ccc}
\alpha & \beta & \gamma - \alpha \\
 x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

Thus, we have \( F(P, (x, y, z)) = z \).

**Proposition 3.4.** If a social choice function \( F \) works monotonically well on a diverse domain \( D \), then the majority rule \( F^M \) also works monotonically well on that domain.

**Proof.** Let us suppose indirectly, that there exists a social choice function \( F \), which works monotonically well on a domain \( D \), whereas \( F^M \) does not. Thus \( D \) contains a cycle for some \( x, y, z \). As a consequence of GD of \( F \), there exists an integer \( n \) (choose, for example, a sufficiently large prime for \( n \)), such that the outcome of any profile, consisting of

\[
\begin{array}{ccc}
 x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

The two groups are divided according to their parity as if seen as permutations. The two groups are called permutation cycles.
n or less preferences associated with equally sized sets of voters, is a singleton.\(^9\)

As \( D \) is diverse, beyond the cycle of \( x, y, z \), we can also find a preference from the other cycle of opposite parity. First, pick the case, when:

\[
\begin{align*}
\text{x} & \quad \text{y} & \quad \text{z} \\
\text{y} & \quad \text{z} & \quad \text{x} \\
\text{z} & \quad \text{x} & \quad \text{y}
\end{align*}
\]

Let us consider the profile \( P \):

\[
\begin{align*}
\alpha & \lesssim \beta \lesssim \gamma \\
x & \quad y & \quad z \\
y & \quad z & \quad x \\
z & \quad x & \quad y
\end{align*}
\]

where \( n\alpha, n\beta, n\gamma \in \mathbb{Z} \) and \( \alpha + \beta + \gamma = 1 \). According to the Lemma 3.3 \( \alpha \lesssim \beta \lesssim \gamma \) holds. (\( \{ P, \{ \text{x}, \text{y}, \text{z} \} \} = \mathcal{Z} \)).

If \( \gamma = \frac{n+1}{2n} \), then for the following profile

\[
\begin{align*}
\text{n} - 1 & \quad \text{n} + 1 \\
\frac{2n}{2n} & \quad \frac{2n}{2n} \\
\text{z} & \quad \text{x} \\
\text{x} & \quad \text{y} \\
\text{y} & \quad \text{z}
\end{align*}
\]

the outcome must be \( z \), according to the M of \( F \). Furthermore, for the profile

\[
\begin{align*}
\text{n} - 1 & \quad \text{n} + 1 \\
\frac{2n}{2n} & \quad \frac{2n}{2n} \\
\text{z} & \quad \text{x} \\
\text{x} & \quad \text{y} \\
\text{y} & \quad \text{z}
\end{align*}
\]

the outcome is again \( x \), according to \( N \).

In addition, for \( P = \)

\[
\begin{align*}
\text{n} - 1 & \quad \text{n} + 1 \\
\frac{2n}{2n} & \quad \frac{2n}{2n} \\
\text{x} & \quad \text{z} \\
\text{z} & \quad \text{y} \\
\text{y} & \quad \text{x}
\end{align*}
\]

the outcome is chosen by \( N \).

If \( \gamma = \frac{n+1}{2n} \) and \( \alpha \leq \beta \leq \gamma \), then in an analogous way we can arrive from profile:

\[
\begin{align*}
\text{n} + 1 & \quad \text{n} + 1 \\
\frac{2n}{2n} & \quad \frac{2n}{2n} \\
\text{y} & \quad \text{z} \\
\text{z} & \quad \text{x} \\
\text{x} & \quad \text{y}
\end{align*}
\]

to profile:

\[
\begin{align*}
\text{n} + 1 & \quad \text{n} + 1 \\
\frac{2n}{2n} & \quad \frac{2n}{2n} \\
\text{z} & \quad \text{x} \\
\text{x} & \quad \text{z} \\
\text{y} & \quad \text{y}
\end{align*}
\]

with outcome \( x \) chosen by \( F \). For profile \( P^* \):

\[
\begin{align*}
\frac{n + 1}{2n} & \quad \frac{n - 1}{2n} \\
\text{x} & \quad \text{z} \\
\text{z} & \quad \text{x} \\
\text{y} & \quad \text{y}
\end{align*}
\]

we get \( z \), according to \( N \). Thus a comparison of profiles \( P \) and \( P^* \) yields a contradiction because of \( A \).

In an analogous way we obtain a contradiction even if we pick other orders outside of the cycle instead of \( (x, z, y) \).

\( \Box \)

4. Concluding remarks

Examining the case of monotonicity, fits into the path followed in the literature in case of the plurality rule (Sanver, 2008; Sanver, 2009) and the Borda rule (Barbie et al., 2006; Puppe and Tasnádi, 2008). The appropriate domains for these rules were examined in the view of IIA, then of non-manipulability and then of monotonicity (leading to Nash-implementability). For both of the plurality and Borda rules the set of appropriate monotonic domains were larger than the set of appropriate non-manipulable domains.

With this paper the same path was completed by a comparison of the rules in the framework of Dasgupta and Maskin (2008a). It could be confirmed again that monotonic domains are not identical to non-manipulable domains; monotonic domains are larger and for different rules, they intersect each other (as for the Borda rule and majority rule). An additional diversity requirement is needed regarding admissible preferences to ensure the superiority of the majority rule.

References


\( ^9 \) As a quick explanation, if we have \( n \) (not necessarily different) preferences associated with equally sized sets of voters, then the possible ratios of votes can be written in the form of \( \frac{\alpha}{n} \) where \( k, \ell \in \mathbb{N}, k + \ell = n \). Thus if \( n \) is a prime greater than any number in \( \left\{ p + q: \frac{p}{q} = 5 \right\} \), then \( S \) would not include any of these ratios.

\( ^{10} \) To show the contradiction for \((z, y, x)\) and for \((y, x, z)\), pick the following profiles

\[
\begin{align*}
\alpha & \lesssim \beta \lesssim \gamma \\
\text{y} & \quad \text{z} & \quad \text{x} \\
\text{z} & \quad \text{x} & \quad \text{y} \\
\text{x} & \quad \text{z} & \quad \text{y}
\end{align*}
\]